Geometric progression as a solution to the extremum problem

Areshev Dmitry
State Budgetary Higher Educational Institution Moscow Region «University of Technology» (UNITECH), Korolev city, Moscow region, 141070, Russia

Samarova Svetlana
Moscow Institute of Physics and Technology (State University), Dolgoprudny city, Moscow region, 141701, Russia

Abstract

The article is devoted to formulation and solving a problem of finding extremums of multivariable special functions. We will demonstrate that the solution of the problem is geometric progression.

Keywords: *multivariable function, extremum, geometric progression.*

1. Formulation of the problem

For any two positive integers a and b, which satisfy inequality a<b and for any natural number n we have to find numbers $x_1, x_2, \ldots, x_n$ satisfying inequalities:

$$a < x_1 < x_2 < \ldots < x_n < b$$

and such numbers that function of *n* variables

$$w = \left(1+\frac{x_1}{a}\right)\left(1+\frac{x_2}{x_1}\right)\cdots\left(1+\frac{b}{x_n}\right) \qquad (1)$$

takes the minimum value.

2. The solution.

Let us introduce the following notation:

$a=x_o$, $b=x_{n+1}$,

and then formula (1) will have the following view:

$$w = \left(1+\frac{x_1}{x_0}\right)\left(1+\frac{x_2}{x_1}\right)\cdots\left(1+\frac{x_{n+1}}{x_n}\right) = \prod_{i=0}^{i=n}\left(1+\frac{x_{i+1}}{x_i}\right),$$

In this formula, we denote a product of factors with the equal structure by the symbol:

$$\prod_{i=0}^{i=n}\left(1+\frac{x_{i+1}}{x_i}\right)$$

Now we will find the critical points [1] of function *w* so that to find the minimum of the function. For this purpose, we set to zero the first partial differentiations of function *w* when

k=0,1,... ,n+1:

$$\frac{\partial w}{\partial x_k} = \frac{\partial}{\partial x_k}\left[\left(1+\frac{x_k}{x_{k-1}}\right)\left(1+\frac{x_{k+1}}{x_k}\right)\right]\prod_{i=0,\, i\neq k-1,\, i\neq k}^{i=n}\left(1+\frac{x_{i+1}}{x_i}\right) = 0 \leftrightarrow$$

$$\leftrightarrow \frac{\partial}{\partial x_k}\left[\left(1+\frac{x_k}{x_{k-1}}\right)\left(1+\frac{x_{k+1}}{x_k}\right)\right] = \frac{1}{x_{k-1}}\left(1+\frac{x_{k+1}}{x_k}\right) + \left(1+\frac{x_k}{x_{k-1}}\right)\left(-\frac{x_{k+1}}{x_k^2}\right) = 0 \leftrightarrow$$

$$\leftrightarrow \frac{1}{x_{k-1}}\left(1+\frac{x_{k+1}}{x_k}\right) = \left(1+\frac{x_k}{x_{k-1}}\right)\frac{x_{k+1}}{x_k^2} \leftrightarrow$$

$$\leftrightarrow \frac{1}{x_{k-1}} + \frac{x_{k+1}}{x_{k-1}x_k} = \frac{x_{k+1}}{x_k^2} + \frac{x_{k+1}}{x_{k-1}x_k} \leftrightarrow$$

$$\leftrightarrow \frac{1}{x_{k-1}} = \frac{x_{k+1}}{x_k^2} \leftrightarrow$$

$$\leftrightarrow x_k^2 = x_{k-1}\, x_{k+1} \qquad (2)$$

Formula (2) is called a characteristic property of geometric progression [2]. It follows from the formula (2) that the numbers

$$x_o, x_1, x_2, \ldots, x_n, x_{n+1} \qquad (3)$$

generate geometric progression if they are put in that order.

Let us denote the ratio of geometric progression (3) by the letter *q*: the formula of general term of geometric progression is the following

$$x_k = x_o q^k, \quad k=0, 1, 2, \ldots, n+1.$$

To receive the formula of the ratio of geometric progression, note that

b=aq^(n+1),

therefore

$$q = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

So we have proved that geometric progression (3) is the critical point of function (1). Now we will prove that the minimum of function w is really implemented in geometric progression (3). For this purpose suppose there are numbers

$$y_o, y_1, y_2, \ldots, y_n, y_{n+1}, \qquad (4)$$

- which not generate geometric progression if

they are put in that order
- and they satisfying inequalities
$y_o < y_1 < y_2 < ... < y_n < y_{n+1}$,

- and satisfying equalities
$a = y_o$, $b = y_{n+1}$

- and this numbers also implement the minimum of function *w*:

$$w = \left(1 + \frac{y_1}{y_0}\right)\left(1 + \frac{y_2}{y_1}\right)\cdots\left(1 + \frac{y_{n+1}}{y_n}\right) = min$$

Since the numbers under consideration (4) do not generate geometric progression, therefore there is such a number *i* which makes the following formula true:

$$\frac{y_i}{y_{i-1}} \neq \frac{y_{i+1}}{y_i}$$

Let us prove that if the both numbers

$$\frac{y_i}{y_{i-1}} \quad and \quad \frac{y_{i+1}}{y_i}$$

were replaced with the following

$$\sqrt{\frac{y_i}{y_{i-1}} \bullet \frac{y_{i+1}}{y_i}} = \sqrt{\frac{y_{i+1}}{y_{i-1}}},$$

then the value of function w would decrease. Actually, it follows from inequality about arithmetic middling and geometric middling [3]: the following formula is true

$$\left(1 + \sqrt{\frac{y_{i+1}}{y_{i-1}}}\right)^2 = 1 + 2\cdot\sqrt{\frac{y_{i+1}}{y_{i-1}}} + \frac{y_{i+1}}{y_{i-1}} \leq$$
$$\leq 1 + \frac{y_i}{y_{i-1}} + \frac{y_{i+1}}{y_i} + \frac{y_i}{y_{i-1}} \bullet \frac{y_{i+1}}{y_i} =$$
$$= \left(1 + \frac{y_i}{y_{i-1}}\right)\left(1 + \frac{y_{i+1}}{y_i}\right),$$

only when

$$\frac{y_i}{y_{i-1}} = \frac{y_{i+1}}{y_i}$$

This conflict shows that it does not exist the set of numbers
$y_o, y_1, y_2, ..., y_n, y_{n+1}$,
different from geometric progression
$x_o, x_1, x_2, ..., x_n, x_{n+1}$
which implements the minimum of function *w*.
The solution to the extremum problem is completed.

3. The answer.

The numbers
a,x1,x2, ..., xn, b
generate geometric progression if they are put in that order.
The ratio of geometric progression can be find by the formula

$$q = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

References

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